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A Stefan Problem with Surface Tension as the Sharp Interface Limit of a Nonlocal System of Phase-Field Type

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A model for the evolution of phase boundaries reminiscent of the phase-field model is considered. The equation related to conservation of thermal energy is diffusive and coupled to an equation for the order parameter, which contains a nonlinear convolution operator, related to the limit of an interacting particle model with Kac-potential. Under diffusive rescaling the solutions converge to solutions of the Stefan problem with kinetic undercooling and surface tension.

KEY WORDS: Stefan problem with surface tension; sharp interface limit; Kac-potential; nonlocal phase-field system.

1. INTRODUCTION

1.1. Structure of the Problem

We understand the equations considered in this paper as a (simplified) model for a liquid undergoing solidification, which is closely related to a stochastic process, see Section 1.2.

The substance undergoing solidification is described by two real valued fields, u and m. The non-conserved field $u(t, x)$ stands for the deviation of the local temperature from the equilibrium temperature T_{eq} at which none of the phases is preferred, i.e., $u = (T - T_{\mathrm{eq}})/T_{\mathrm{eq}}$, and $m(t, x) \in [-1, 1]$ is the order parameter which describes how close the material is to its solid $(-)$ or liquid $(+)$ state. Here t is time and x is a point in $\mathbb{R}^n$. (We will assume periodic boundary conditions.) The local energy density $e(u, m)$ is such that its integral is conserved. We assume for

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simplicity $e = u + m$. The energy e is the sum of the thermal energy (linear in u) and the energy "stored" in the structure of the two phases, the *latent heat*.

The evolution is governed by the *nonlocal phase field equations* with *periodic boundary conditions*, where λ is a small parameter:

$$\lambda^2 \partial_t m^\lambda(t, x) = -m^\lambda(t, x) + \tanh(h_A(t, x)), \tag{1.1}$$

$$h_A(t, x) := \beta \left[\int_\Omega \lambda^{-n} J\left(\frac{x-y}{\lambda}\right) m^\lambda(t, y)\, dy + \lambda u^\lambda(t, x) \right],$$

$$\partial_t (u^\lambda(t, x) + m^\lambda(t, x)) = \Delta u^\lambda(t, x). \tag{1.2}$$

Here $J \geqslant 0$ (ferromagnetic), $\int_{\mathbb{R}^n} J(x)\, dx = 1$, $J(x)$ depends only on $|x|$ (isotropic), and $\beta > 1$. A Lyapunov functional for the system is given by the free energy

$$F^\lambda(m, u) := \frac{1}{4} \iint \lambda^{-n} J\left(\frac{x-y}{\lambda}\right) (m(x) - m(y))^2\, dx\, dy$$

$$+ \int_\Omega W_\beta(m(x))\, dx + \frac{\lambda}{2} \int_\Omega u^2(x)\, dx \tag{1.3}$$

$$W_\beta(m) := \frac{1}{\beta} \left[\frac{1-m}{2} \ln\left(\frac{1-m}{2}\right) + \frac{1+m}{2} \ln\left(\frac{1+m}{2}\right) \right] - \frac{1}{2} m^2 - c_\beta, \tag{1.4}$$

where the normalization c_β is chosen such that $\min_{\mathbb{R}} W_\beta(m) = 0$. For $\beta > 1$ the function W_β is a double-well potential with two distinct minimizers $\pm m_\beta$, the two distinct nonzero solutions of $m_\beta = \tanh(\beta m_\beta)$. (For $\beta \leqslant 1$ the unique minimizer is 0.)

The parameter β stands in statistical physics for an inverse temperature, and in our model u is related to the temperature, so the question arises how they are related. We skip the issue here and refer the reader to Section 1.2, where modeling issues are addressed.

Note the competition of the double-well potential W, enforcing $m \sim \pm m_\beta$, and the nonlocal J-dependent interaction term which penalizes the transition from $+m_\beta$ to $-m_\beta$. The Γ-convergence of related nonlocal functionals was studied, e.g., in refs. 1 and 2.

Equations (1.1), (1.2), and their Lyapunov functional should be compared with the (local) phase field equations

$$\partial_t m = \Delta m - \frac{1}{\lambda^2} (W'(m) + \lambda u) \tag{1.5}$$

$$\partial_t (u + m) = \Delta u, \tag{1.6}$$

($W(m)$ is again a double-well potential) and the corresponding functional

$$\tilde{F}(m) := \int \lambda \, |\nabla m|^2 \, dx + \frac{1}{\lambda} \int W(m) \, dx.$$

The relation between the two systems can be understood informally by noting that in Fourier space $\lambda^{-2}\widehat{(J^\lambda * m - m)}(\xi) = [c_J \, |\xi|^2 + O(\lambda^2 \, |\xi|^4)] \, \hat{m}(\xi)$. ($J^\lambda(\cdot) = \lambda^{-n} J(\frac{\cdot}{\lambda})$)

When $\lambda \to 0$ in (1.1) one can guess that m should quickly relax to be close to one of the spatially constant equilibria $\pm m_\beta$ of the uncoupled system.

Moreover the motion of the transition layer from $-m_\beta$ to $+m_\beta$ should be slow on the time scale where $\lambda = O(1)$. So we assume that its profile in the fast direction, i.e., the direction normal to the interface separating the $+m_\beta$ from the $-m_\beta$ region, looks at first order like a one dimensional stationary solution of (1.1), connecting $+m_\beta$ and $-m_\beta$. It is known (see refs. 1, 2) that in 1-d there is a strictly increasing, antisymmetric function $\bar{m}(r) \in C^\infty$ connecting $\pm m_\beta$, called *instanton*. $\bar{m}(r)$ solves

$$0 = -\bar{m}(r) + \tanh\left(\beta \int_{\mathbb{R}} \bar{J}(r - r') \, \bar{m}(r') \, dr'\right), \qquad \lim_{r \to \pm\infty} \bar{m}(r) = \pm m_\beta, \quad (1.7)$$

$$\bar{J}(r) = \int J(|(r, y_1, \ldots, y_{n-1})|) \, \mathrm{d}y_1 \cdots \mathrm{d}y_{n-1}. \quad (1.8)$$

(This means $\bar{J}$ is the effective kernel if the convolution operator is applied to functions depending on one variable only.)

Let $\Sigma(t)$ be the boundary of $\Omega^-(t)$, which is the set where $m(t) \to -m_\beta$ (solid). We assume that $\Omega^- \subset \Omega^+$, i.e., a solid droplet forming in a liquid. We make the ansatz

$$m^\lambda(t, x) = \bar{m}\left(\frac{\mathrm{d}(x, \Sigma(t))}{\lambda}\right) + \lambda m_0\left(t, x, \frac{\mathrm{d}(x, \Sigma(t))}{\lambda}\right) + O(\lambda^2),$$

$$u^\lambda(t, x) = u(t, x) + O(\lambda),$$

where d is the signed distance, negative in the solid.

The formal ansatz leads to the expectation that to highest order (Σ, u) solve a free boundary problem sometimes referred to as Mullins–Sekerka problem with kinetic undercooling, which we call here *Stefan problem with surface tension and kinetic undercooling*:

$$V(t,x) = -\theta\kappa(t,x) - 2Nm_\beta \cdot u(t,x) \qquad \text{on} \quad \Sigma(t), \tag{1.9}$$

$$\partial_t u = \Delta u \qquad \text{on} \quad \text{int}(\Omega \setminus \Sigma(t)) \tag{1.10}$$

$$[\nabla u \cdot \nu]_\Sigma = 2m_\beta V \qquad \text{on} \quad \Sigma(t), \tag{1.11}$$

where $[f]$ denotes the jump in normal direction (from solid to liquid) of f across Σ, κ is the mean curvature (positive for convex droplet) and V the normal velocity of Σ. Observe that (1.10) and (1.11) contain the conservation of the total energy.

We remark that the coefficient θ depends on certain "tangential moments" of the kernel, and N is a normalization for the instanton in a weighted L^2-space. The appearance of θ is a special feature of the nonlocal evolution not shared by the local phase field equations ($N = 2\theta = m_\beta = 1$).

It is the aim of this paper to make this ansatz rigorous under certain conditions (developed interface) and as long as the free boundary problem has a classical solution.

We would like to mention that we are aware of work by Carlen, Carvalho, and Orlandi on a similar theorem for a nonlocal version of the *Cahn–Hilliard* equation.[28]

The existence of classical solutions local in time for this free boundary problem has been shown by X. Chen and F. Reitich,[13] if u has Dirichlet or Neumann boundary conditions.

1.2. Modeling Issues: A Stochastic Perturbation of a Free Boundary Problem

In refs. 18–20 a nonlocal evolution equation similar to (1.1) is derived from an interacting particle model, where the parameter β stands for the inverse temperature.

If we see the system (1.1)–(1.2) as a model for solidification and u as the deviation from the temperature T_{eq} at which both phases have equal free energy, then what is β? Technically we could justify the assumption that β is constant by starting with $\beta(T)$ and then assuming that the deviation from T_{eq} is small, i.e., $\beta \approx \beta(T_{\text{eq}})$.

Of course it is questionable whether a double-well potential with two symmetric phases can describe solidification, a situation where the phases have different degrees of order and a higher temperature favors the less ordered phase. So we prefer to assume that β is just a parameter unrelated to u and view the model simply as a perturbation of the free-boundary value problem (1.9)–(1.11).

In fact our motivation to study the nonlocal phase field equations (1.1) and (1.2) was the close relation between them and a system of equations which is itself the limit of a stochastic process with Kac-potentials and Glauber dynamics modeling phase change in the presence of a diffusing external field. Thus this stochastic process can be interpreted as a small stochastic perturbation of the free boundary problem, which could be used to understand effects like nucleation and formation of "mushy regions." In order to present the reader with some motivation, we describe these ideas more precisely.

The Stefan problem without surface tension, i.e., the system consisting of (1.10) and (1.11) coupled with $u = 0$ on $\Sigma(t)$ admits *weak* solutions which develop a so-called mushy region, i.e., a region where the system is not clearly in one of the two phases, see e.g., ref. 30, IV.3. (The state of the system can be interpreted as infinitesimally fine mixture of the two phases.)

If surface tension is considered, then the condition "$u = 0$ on $\Sigma(t)$" is replaced by an equation containing the mean curvature (first variation of the surface energy), e.g., (1.9) or the Stefan-problem with Gibbs–Thomson law, $u = \theta\kappa$. Weak solutions for these models (see, e.g., refs. 26 and 29) do not develop mushy regions, and therefore they do not converge always to the weak solutions of the Stefan problem without surface tension as the surface tension (i.e., the parameter θ) vanishes. A possible explanation might be that the mushy region is due to many *nucleation* events occurring simultaneously at different sites, so a stochastic perturbation of a phase-field approximation to the Stefan problem should be considered. This could be done by adding noise to the local phase field system, i.e., keeping energy conservation (1.6) and adding the noise term $\epsilon dB(t, x)$ to the order parameter equation, where B is white in time and (in higher space dimensions) sufficiently regular in space. For this approach see refs. 5 and 6.

Here we model the phase change by spatially discrete random variables (spins) $\sigma(t, x) \in \{-1, +1\}$, where x is a site on a periodic integer lattice. The spins flip at rates depending both on the average of their neighbors via a Kac-potential and on u. We hope that our approach, though a caricature, is more natural because the driving mechanism of phase change phenomena is actually collective behavior of interacting particles.

More precisely, the flip rate c (infinitesimal probability of sign change) for the spin at site x is a function of

- average of neighbor spins (surface energy)

$$h(x, \boldsymbol{\sigma}) := \sum_y \gamma^n J(\gamma\, |x - y|)\, \sigma(y)$$

- (averaged) field $\lambda\tilde{u}(\gamma x)$, where $\tilde{u}(x) := (K^{\lambda^2} * u)(x)$, $0 < \lambda \ll 1$, $K \in C_0^\infty$, $K^{\lambda^2}(z) := \lambda^{-2n} K(\lambda^{-2} z)$

$$c(x, \sigma(t, x), \boldsymbol{\sigma}(t)) := \tfrac{1}{2}\{1 - \tanh[\beta\sigma(t, x)(h(x, \boldsymbol{\sigma}) + \lambda\tilde{u}(t, \gamma x))]\}.$$

The equation for the field u (defined on the torus) is given by the conservation of total energy:

$$\partial_t u = \Delta u - \partial_t (K^{\lambda^2} * \sigma).$$

The small kernels (of width λ^2) had to be introduced for technical reasons, to overcome the singularity of the heat kernel at $t = 0$.

Now average the spins over a box containing $(\gamma^{-\alpha})^n$ sites (Coarse-Graining):

$$m^\gamma(t, x, \boldsymbol{\sigma}) := (\gamma^\alpha)^n \sum_{|y_i - x_i| < \frac{1}{2}\gamma^{-\alpha},\, 1 \leqslant i \leqslant n} \sigma(t, y).$$

The dependence on σ emphasizes that this is a random variable.

Due to the averaging over many lattice sites as $\gamma \to 0$ in the definition of $h(x, \boldsymbol{\sigma})$ and $\tilde{u}$, these coarse-grained spins converge by a sort of law of large numbers:

The *modified* nonlocal phase field equations are obtained by replacing m in (1.2) by $K^{\lambda^2} * m$, and u in (1.1) by $\tilde{u}$. Now introduce the scaling $\gamma x = x'$ so the interaction range is $O(1)$, whereas time is unscaled.

The random variables $m^\gamma(t, \gamma x, \boldsymbol{\sigma})$ converge as $\gamma \to 0$ in probability to the solution of the modified nonlocal phase field equations: Let $m(t, x')$ be the solution of these modified nonlocal phase field equations, and $\tilde{m}^\gamma(t, x', \boldsymbol{\sigma})$ the piecewise constant extension of the lattice random variable $m^\gamma(t, \gamma x, \boldsymbol{\sigma})$. Then for any $\delta > 0$

$$\mathbb{P}(\sup_{t, x'} |\tilde{m}^\gamma(t, x', \boldsymbol{\sigma}) - m(t, x')| > \delta) \to 0 \qquad \text{as} \quad \gamma \to 0,$$

see ref. 15. The convergence holds for times of order λ^{-2} and for x' on a periodic domain of side length $O(\lambda^{-1})$, where $\lambda \to 0$ but $\gamma \ll \lambda$. So by combining ref. 15 and the present paper one could go directly from the stochastic process to a Stefan problem. (For further explanations of the method see refs. 18 and 24.)

For a far more detailed analysis of the uncoupled system (i.e., u is constant) see ref. 18.

We remark that nonlocal models have been proposed and used for the study of *anisotropic* phase change, using general kernels $J(x-y)$ instead $J(|x-y|)$, see, e.g., refs. 4, 8, and 11.

C. K. Chen and P. C. Fife consider in ref. 11 a class of nonlocal phase field equations and do a formal expansion to find the sharp interface limit for kernels which are anisotropic and temperature dependent.

In these papers the part involving the convolution is linear in the order parameter, i.e., equations of the form $\partial_t m = J * m - m - W'(m)$. The reason for our choice (1.1) is that our equations come from a stochastic process with flip intensities which have to be nonnegative and bounded. To our knowledge there is no stochastic lattice model related to the order parameter equation with linear convolution and a double well potential.

1.3. The Method of the Proof

In the case where there is only the order parameter equation (1.1) with a *constant* field u, the limit free boundary problem is motion by mean curvature. The single nonlocal equation has a comparison principle, i.e., if for two solutions m_1, m_2 initially $m_1(t) \leqslant m_2(t)$, then $m_1(t+s) \leqslant m_2(t+s)$ for all $s \geqslant 0$. This can be used to "squeeze" the actual solution between two approximate solutions derived from the ansatz, see, e.g., in refs. 17 and 18 by A. De Masi, E. Orlandi, E. Presutti, and L. Triolo. Using the comparison principle and a generalized solution (viscosity solution for the level set equation) for the free boundary problem, these results were extended by M. A. Katsoulakis and P. E. Souganidis in refs. 24 and 25 to the anisotropic case and past the appearance of singularities of the free boundary problem. This approach was put in a general framework by G. Barles and P. E. Souganidis.[9] For an approach using geometric measure theory for the local phase field equations see ref. 29.

The system considered in this paper does not have a comparison principle and the methods relying on it do not apply. Our approach follows ideas by N. D. Alikakos, P. W. Bates, and X. Chen in ref. 3 (Cahn–Hilliard equation) and by G. Caginalp and X. Chen (local phase field equations) in ref. 10.

We briefly describe the method at an informal level:

Step 1. Show the existence of a function which solves the equation up to a small right hand side (approximate solution) by matching formal asymptotic expansions for the region around the free boundary and away from the free boundary. The limit free boundary problem is determined by solvability conditions for the asymptotic expansions.

Step 2. Show linear stability for the linearization of the equations around this approximate solution, i.e., the spectrum of the linearized operator $\lambda^{-2}L^{\lambda}$ is bounded from one side uniformly in λ, or $Q^{\lambda}(\Psi) \geqslant -\lambda^2 C \, \|\Psi\|^2$, where $Q^{\lambda}(\Psi) = \langle L^{\lambda}\Psi, \Psi\rangle$.

Step 3. Control the error introduced by the nonlinear terms, get a first estimate using a Gronwall argument and apply regularity theory to improve it: Let Ψ be the difference between the true and the approximate solution. Then we have for some suitable (weighted L^2-) norm $\|\cdot\|$ and the inner product $\langle \cdot, \cdot \rangle$

$$\partial_t \, \|\Psi\|^2 = -\langle \Psi, \lambda^{-2}L^{\lambda}\Psi\rangle + R^{\lambda}(\Psi) \leqslant c \, \|\Psi\|^2 + r^{\lambda}(\|\Psi\|) \, \|\Psi\|^2$$

by step 2 and step 3.

We give some brief remarks on the first two steps:

Approximate Solution. As in ref. 10, the inner expansion is constructed using *multiscale asymptotic expansions*: We treat the direction orthogonal to the limit interface (fast variable) as additional independent variable, which simplifies computations, but for consistency we have to introduce additional unknown functions, comparable to the appearance of Lagrange multipliers in calculus of variations.

Stability of the Linearization. The main difficulty for the linear stability in our case is to derive from the stability of the instanton in one dimension, shown e.g., in ref. 19, the stability for the linearization around the approximate solution in several dimensions: In principle, tangential oscillations could grow. Here we cannot follow the methods used for the Laplacian by X. Chen in ref. 12, as we cannot split the kernel into a convolution in tangential and in normal direction, whereas expressions like $|\nabla\Psi|^2$ split. However Chen gets much stronger results, which also apply to the sharp interface limit of the Cahn–Hilliard equation.

Here we use the strict monotonicity ($\bar{m}' > 0$), the decay properties of $\bar{m}'(r)$ as $r \to \pm\infty$ and the symmetry of the instanton to construct a function Φ, which solves up to an error of smaller order the equation for the eigenfunction for an eigenvalue of order λ^2, and which is strictly positive close to the interface. Close to the interface we can consider $\Psi\Phi^{-1}$ to show that the infimum of Q^{λ} is of order λ^2. Away from the interface we use the fact that the double-well potential $W(m)$ is convex there and a very rough a-priori estimate obtained by comparing a function with its "tangential average."

1.4. Results

Let $J(x) = J(|x|)$ be an isotropic, *nonnegative* (ferromagnetic) interaction kernel in $C^1(\mathbb{R}^n)$ with support in $\{x: |x| \leqslant 1\}$, $\beta > 1$, and $\int_{\mathbb{R}^n} J(x)\,\mathrm{d}x = 1$.

Assume that the free boundary problem (1.9)–(1.11) has a classical solution on $[0, T_{\max})$ for regular initial data $u(0, x) = g$, $\Omega^{\pm}(0) = \Omega_0^{\pm}$ and periodic boundary conditions.

Further we assume that the free boundary $\Sigma(t)$ is a smooth boundary of a solid "droplet" in a liquid environment (solidification), i.e., $\Omega^-(t) \subset \Omega^+(t) \subset \Omega$ and $\partial\Omega^-(t) = \Sigma(t)$ for all $t \in [0, T_{\max})$. (In particular Σ does not touch the periodic (fixed) boundary of the domain.)

We will show that the actual solutions of the nonlocal phase field system stay close to the approximate solutions given by the following theorem:

Theorem 1.1.

(1) For any integer $k \geqslant 1$ there are functions m_a^λ, u_a^λ such that they solve system (1.1)–(1.2) with periodic boundary conditions up to a right hand side r^λ in (1.1) such that $\|r^\lambda\|_\infty < \lambda^k$, whereas (1.2) is solved exactly.

Moreover let $\Sigma(t) = \partial\Omega^-(t)$ such that $(\Sigma(t), u(t))$ is a smooth solution for the Stefan problem (1.9)–(1.11) with initial data as above. Then $u_a^\lambda(t) = u(t) + O(\lambda)$ and $m_a^\lambda(t) = \pm m_\beta + O(\lambda)$ in $\Omega_0^{\pm}(t)$, if $|\mathrm{dist}(x, \Sigma(t))| > C\lambda\,|\ln(\lambda)|$ for some $C > 0$.

(2) Further for $k > p(n)$, (n is the space dimension) we have for the true solutions $(m_{\text{true}}^\lambda, u_{\text{true}}^\lambda)$ of (1.1)–(1.2) starting from the same initial values: $\|m_a^\lambda - m_{\text{true}}^\lambda\|_{L^\infty} + \|u_a^\lambda - u_{\text{true}}^\lambda\|_{L^\infty} \to 0$ as $\lambda \to 0$.

As the approximate solutions for the order parameter m are close to $\pm m_\beta$ in $\Omega^{\pm}(t)$, we get immediately the following corollary:

Corollary 1.2 (Sharp Interface Limit for Developed Interfaces). For any $\delta > 0$ the solutions of (1.1)–(1.2) with initial conditions as in Theorem 1.1 (i.e., a developed interface) are such that $u^\lambda(t) \to u(t)$ uniformly in the time interval $[0, T_{\max} - \delta)$ and $m^\lambda(t) \to \pm m_\beta$ in $\Omega^{\pm}(t)$ a.e. and in L^1.

1.5. Notation

n denotes the space dimension and λ will be the small parameter. $|x|$ denotes the Euclidean norm in $\mathbb{R}^n$.

Convention on Constants. C, c, etc. denote (not necessarily the same) constants which do not depend on λ.

By a slight abuse of notation we write $J(x) = J(\sqrt{x_1^2 + \cdots + x_n^2})$, where the function J on the left hand side is a function on $\mathbb{R}^n$ and the J on the right hand side is a function on $\mathbb{R}$, denoted by J as well. $\bar{J}$ is as in (1.8).

$$J^\lambda(x-y) = \lambda^{-n} J\left(\frac{x-y}{\lambda}\right), \qquad \bar{J}^\lambda(r-r') = \lambda^{-1}\bar{J}\left(\frac{r-r'}{\lambda}\right).$$

$$N^{-1} := \int_{\mathbb{R}} \frac{(\bar{m}'(r))^2}{1-\bar{m}(r)^2}\,\mathrm{d}r, \tag{1.12}$$

$$\theta := \frac{N}{2}\iiint \bar{m}'(z)\, J\left(\hat{z}^2 + \sum_{k=2}^{n} y_k^2\right) \bar{m}'(z+\hat{z}) \left(\sum_{k=2}^{n} y_k^2\right) \mathrm{d}\hat{z}\,\mathrm{d}z \prod_{k=2}^{n} \mathrm{d}y_k.$$

N is the normalization of $\bar{m}'$ with respect to a measure depending on $\bar{m}$, and θ a tangential second moment of the kernel.

For $x \in \mathbb{R}^n$ let $r(x) := \operatorname{dist}(x, \Sigma)$, where dist is the *signed* distance, positive in Ω^+.

We will sometimes use $r^\lambda(x)$ to indicate that we mean the distance from the zero level set of the approximate solution.

Denote by $\mathrm{Pr}_\Sigma\colon \{x\colon |\operatorname{dist}(x, \Sigma)| < \delta\} \to \Sigma$ the projection to Σ. (It is well defined for δ small.)

Let for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \sum_{i=1}^n \alpha_i$, $y^\alpha = \prod_{i=1}^n y_i^{\alpha_i}$, and $D^\alpha f(x)$ stands for the coefficient of y^α in the Taylor expansion of f at x.

For a function $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ let $f'(x, r) := \frac{\partial}{\partial_r} f(x, r)$.

We will write $\|f\|_2$ for the L^2-norm, and $\|f\|_\infty$ for the L^∞-norm.

$[f(\lambda) \mid i]$ denotes the ith order term if the quantity in brackets is formally expanded, i.e., $f(\lambda) = \sum_i [f(\lambda) \mid i]\, \lambda^i$.

2. ASYMPTOTIC EXPANSION

In this section we will prove Theorem 1.1 (1) by the method of matched asymptotic expansions. We construct different expansions away from the free boundary and close to the free boundary, where the order parameter changes quickly in the direction orthogonal to the free boundary. The idea is to use Taylor expansion in the slow directions to end up with a one-dimensional convolution equation in the fast direction.

2.1. Outer Expansion

We seek a solution of the system in a neighborhood of one of the stable equilibria, either of $+m_\beta$ or of $-m_\beta$. There the system is stable, small perturbations in any direction are damped, so a reasonable ansatz is $m^\lambda(t,x) = m_\beta + \mathcal{O}(\lambda)$. We look for an expansion for the outer solution in a formal power series in λ, i.e.,

$$m^{\lambda,\pm}(t,x) = \pm m_\beta + \sum_{i=0}^{k} \lambda^{i+1} m_i^{\pm}(t,x), \qquad u^{\lambda,\pm}(t,x) = \sum_{i=0}^{k} \lambda^i u_i^{\pm}(t,x).$$

In order to determine the (m_i, u_i), we use the Taylor-expansion for m and replace the convolution by multiplication of derivatives with moments of the kernel, so at the end we get an algebraic equation.

Let $(\mathbf{M}_\alpha(J)) := \int_{\mathbb{R}^n} (\prod_{l=1}^n x_l^{\alpha_l})\, J(x)\, \mathrm{d}x$ be the α-moment of the kernel J for the multi-index α. Then $\mathbf{M}_{(0,\ldots 0)} = 1$, $\mathbf{M}_\alpha = 0$ for $|\alpha| = 1$, and formally

$$(J^\lambda * f)(x) = \sum_{i=0}^{\infty} \sum_{|\alpha|=i} \mathbf{M}_\alpha(J)(D^\alpha m)(x).$$

Thus the *formal* outer expansion equations read for Eqs. (1.1)–(1.2)

$$\partial_t u_i^{\pm} - \Delta u_i^{\pm} = -\partial_t m_{i-1}^{\pm} \tag{2.1}$$

$$m_i^{\pm} = \frac{\beta(1-\bar{m}_\beta^2)}{1-\beta(1-\bar{m}_\beta^2)} u_i^{\pm} + \frac{1}{1-\beta(1-\bar{m}_\beta^2)} (-\partial_t m_{i-2}^{\pm} + B^{i-1}),$$

$$B^{i-1} = B^{i-1}(\{\mathbf{M}_\alpha(J)\, D^\alpha m_{i-k}^{\pm}\}_{|\alpha|=k,\, 2\leqslant k\leqslant i}, \{u_k^{\pm}\}_{k\leqslant i-1}). \tag{2.2}$$

2.2. Inner Expansion

We seek a formal solution of the form

$$m^\lambda(t,x) = \bar{m}\left(\frac{d^\lambda(t,x)}{\lambda}\right) + \sum_{i=0}^{\infty} \lambda^{i+1} m_i\left(x, \frac{d^\lambda(t,x)}{\lambda}\right), \tag{2.3}$$

$$u^\lambda(t,x) = \sum_{i=0}^{\infty} \lambda^i u_i\left(x, \frac{d^\lambda(t,x)}{\lambda}\right) \tag{2.4}$$

$$d^\lambda(t,x) = \sum_{i=0}^{\infty} \lambda^i d^i(t,x), \tag{2.5}$$

$$|\nabla d^\lambda(t,x)| = 1 \tag{2.6}$$

where $m_i, u_i \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ with all derivatives bounded and $d^i \in C^k(\mathbb{R}^n)$, $d^0 = \mathrm{dist}(x, \Sigma^0(t))$. (*Signed* distance.) Here Σ^0 is the free boundary of the limit Stefan problem. Of course later the expansion will be truncated at some large integer K. For the necessity of corrections to the distance from the limit free boundary, i.e., $d^0 + \lambda d^1 + \cdots$, see also refs. 3, 9, 10, and 24. (Instead of expanding the distance function one could as well seek a solution of the form $\bar{m}(\frac{h^\lambda+d^0}{\lambda})+\sum_i \lambda^i m_i(\frac{d^0}{\lambda})$.) Note that (2.6) is equivalent to the so-called *distance equations*:

$$|\nabla d^0|^2=1, \quad \nabla d^0\,\nabla d^i=-\tfrac{1}{2}\sum_{j=1}^{i-1}\nabla d^{i-j}\,\nabla d^j, \qquad i\geqslant 1. \tag{2.7}$$

Matching Conditions. We need to impose matching conditions for $z\to\pm\infty$ uniformly in (t,x). We require convergence of all derivatives with an exponential rate, up to some order depending on the required quality of the approximate solution, see refs. 3 and 10.

Further Remarks. We remark that an approach as in ref. 10 will lead to the appearance of very high derivatives in the "lower order parts" of the equation, because unlike a Laplace equation, which involves only second derivatives, expanding the convolution operator around the one-dimensional convolution involves in principle the whole Taylor expansion either of the kernel or of the unknown functions. In both cases we need to restrict the class of admissible kernels. Our choice is not optimal, but we do not strive for generality here.

Denote by $L^{(1)}_{\bar{m}}=f$ the linearization of the one-dimensional integral operator in (1.7) around the instanton (standing wave) $\bar{m}$. $L^{(1)}_{\bar{m}}=f$ has a solution if and only if f is orthogonal to $\bar{m}'$ in a weighted L^2 norm. (Fredholm alternative.) More precisely, the function space is L^2 with respect to the measure $\mu(\mathrm{d}x):=N\frac{\mathrm{d}x}{\beta(1-\bar{m}^2)}$, where N is a normalization such that $\|\bar{m}'\|^2_{\mu(\mathrm{d}x)}=1$. (See Notation and Appendix.) This fact concerning the one-dimensional situation enters in the following way: We use Taylor expansion in the tangential direction and reduce the equations to the form $L^{(1)}_{\bar{m}}m_j(t,x,z)=R^{j-1}(t,x,z)$, where the operator acts on functions of the variable $z\in\mathbb{R}$ and R^{j-1} depends on already computed quantities and on d^i, u_i. We can find a solution if and only if $R^{j-1}(t,x,z)$ is orthogonal to $\bar{m}'(z)$ in $L^2(\mu(\mathrm{d}x))$. This leads to an evolution equation for d and thus to a free boundary problem. (The boundary is $\partial\{d>0\}$.) However the resulting system for d, m and u will in general be overdetermined: d solves a *linear* PDE with the constraint $|\nabla d|=1$. The reason for this is that we treat z and x as independent variables, although they are related by $\lambda z-d^\lambda(x)=0$. So we follow ref. 10 and introduce new unknown functions $g^\lambda(t,x)$, $h^\lambda(t,x)$, $l^\lambda(t,x)$. They appear in the equations multiplied by the term by $(\lambda z-d^\lambda)$, so they cannot affect the solution in the region we are interested in. The resulting *corrected* system of equations has a unique solution.

The following lemma is about the expansion of a convolution in directions tangential to the interface. Let $z(x):=\lambda^{-1}\,d^\lambda(x)\in\mathbb{R}$. Let $'$ denote derivatives w.r.t z, whereas the operators D^α and ∇ (see 1.5) act only on functions of the variable x.

Lemma 2.1. For $K \in \mathbb{N}$ and a function $f^\lambda(x, z) = \sum_{i=0}^K \lambda^i f_i(x, z)$ and d^λ as in (2.5) solving the distance equations (2.7), define

$$J^\lambda(m, d^\lambda)(x) := J^\lambda * f(x, \lambda^{-1} d^\lambda(x)).$$

Then we get the following formal asymptotic expansion:

$$J^\lambda(f, d^\lambda)(x) = \int \bar{J}(r)\, f_0(x, z(x)+r)\, \mathrm{d}r + \sum_{i=1}^K \lambda^i J_i(f, d^\lambda)(x) + \mathcal{O}(\lambda^{K+1}),$$

$$\begin{aligned} J_i(f, d^\lambda)(x) = &\int \bar{J}(r)\, f_i(x, z(x)+r)\, \mathrm{d}r \\ &+ \frac{\Delta d^{i-1}(x)}{2} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} J(z^2+|y|^2)\, f_0'(x, z(x)+z)\, |y|^2\, \mathscr{H}^{n-1}(\mathrm{d}y)\, \mathrm{d}z \\ &+ \nabla d^{i-1}(x) \cdot \nabla d^1(x) \int \bar{J}(r)\, f_0'(x, z(x)+r)\, r\, \mathrm{d}r \\ &+ R_J^{i-1}(x, z(x)), \end{aligned}$$

where R_J^{i-1} is obtained by convolutions of $\{D^\alpha d^k(x)\}_{k \leqslant i-2,\, |\alpha|+k \leqslant i+1}$ and $\{\partial_z^l D_x^\alpha f_k\}_{k \leqslant i-1,\, |\alpha|+k+l \leqslant i}$ with expressions like $J_p(r) := \int J(r, y)\, P(r, y)\, \mathrm{d}y$, where P is a polynomial.

The idea of the proof is to insert the Taylor expansion of d^λ at x_0 with $\lambda z_0 = d^\lambda(x_0)$ into the Taylor expansion of f around $(x_0, z_0 + \nabla d^0(x_0) \cdot y) = (x_0, z_0 + r)$.

To get the inner expansion equations, we have to expand the nonlinearity (hyperbolic tangent) as well. Unlike in ref. 10, the nonlinearity involves both phase and temperature.

On the left hand side of the order parameter equation (OPE), (1.1) we have to expand $\partial_t m^\lambda(x, \lambda^{-1} d^\lambda(x, t))$.

When taking into account that due to the time rescaling the right hand side of (1.1) is multiplied by λ^{-2}, then we get the *uncorrected* inner expansion equations for the order parameter:

$$B_p^i + B_O^i + B^{i-1} = L_{\bar{m}}^{(1)}\, m_i$$

$$L_{\bar{m}}^{(1)}\, m_i := -m_i + (1 - \bar{m}(z)^2) \int \bar{J}(r) * m_i(x, z+r)\, \mathrm{d}r,$$

$$B_p^i := \bar{m}'(z)(\partial_t d^i(t, x) - \theta\, \Delta d^i(t, x) - \overline{u_i}(t, x, z)),$$

$$\overline{u_i}(t, x) := \int N \bar{m}'(z)\, u_i(t, x, z)\, \mathrm{d}z \tag{2.8}$$

B_O^i is the part of $J_i(m,d)$ depending on u_i, d^i, but projected on the orthogonal complement of span$(\bar{m}')$ with respect to the measure $\mu(\mathrm{d}z)$, B_p^i is the part parallel to $\bar{m}'$. Note that the solvability condition for each level of the asymptotic expansion requires that the right hand side is orthogonal to $\bar{m}'$.

By symmetry arguments one can see that $\int_{\mathbb{R}} \bar{J}(r)\, \bar{m}'(z+r)\, r\, \mathrm{d}r$ is orthogonal to $\bar{m}'(z)$, that is why there is no $\nabla d^1\, \nabla d^i$ in B_p^i. Moreover in the equation for the first order correction to the instanton, m_0, this term disappears totally as $\nabla d^1\, \nabla d^0 = 0$ by the distance equations. (See 2.1)

The transport coefficient θ comes from a projection on $\bar{m}'$, see (1.12). B^{i-1} depends only on already computed terms of the expansion.

Now we follow ref. 10 and add additional unknown functions to the equations for u and m in such a way that the additional terms vanish on the submanifold of interest, $\lambda z = d^\lambda(t,x)$. The strategy is to find first a solution on $\{d^0(t,x)=0\}$, and then to define the additional unknown functions at the respective order by the requirement that the equation is solved for all (t,x,z).

Add to the OPE (1.1) the term $\lambda^{-1} g^\lambda(t,x)\, \eta'(z)(d^\lambda - \lambda z)$, where $g^\lambda(t,x)$ is an additional unknown function and η a function connecting 0 and 1 monotonically s.t. η' serves as cut-off, see ref. 10.

We have to add an expression containing *two* additional unknowns, $h^\lambda(t,x)$ and $l^\lambda(t,x)$, to the equation for u. One of them will be needed for the matching, the other for the solvability (or secularity) condition.

In order to solve the inner expansion equations, we will need to extend the *outer* solutions $u_i^\pm$, $m_i^\pm$ on a small neighborhood of *both* sides of the free boundary. (As the zero level set of d^λ is only λ-close to the free boundary of the Stefan problem.) We can do so such that the order parameter equation is solved exactly, but the energy equation will not be solved exactly: In the domain of extension, there will be a remainder $R_\pm^i$, called discrepancy term.

If we add the same expression as in ref. 10, put $u(t,x,\lambda^{-1}\, d^\lambda(t,x))$ in Eq. (1.2) and use $|\nabla d^\lambda| = 1$, then we derive in the same way as ref. 10 the corrected inner expansion equations:

$$\left[u_i(t,x,z) - \sum_{j=0}^{i} \eta(z)\, d^{i-j}(t,x)\, h^j(t,x) \right]'' = A^{i-1} + A_1^{i-2} + A_2^{i-2},$$

$$\begin{aligned} A^{i-1} &= \partial_t\, d^{i-1} \bar{m}' + (\partial_t d^0 - \Delta d^0)\, u'_{i-1} + (l_{i-1} d^0 + l_0 d^{i-1})\, \eta' - h_{i-1} z \eta''(z) \\ &\quad + (\partial_t\, d^{i-1} - \Delta d^{i-1})\, u'_0 - (\nabla d^0\, \nabla u'_{i-1} + \nabla d^{i-1}\, \nabla u_0{}') \\ A_1^{i-2} &= [\partial_t\, u^\lambda(t,x,z) - \Delta u^\lambda(t,x,z) + \partial_t m^\lambda(t,x,z) \,|\, i-2] \\ &\quad - [R_+^{i-2}(t,x)\, \eta(C_1+z) + R_-^{i-2}(t,x)\, \eta(-C_1-z)]. \end{aligned} \tag{2.9}$$

(Note that this is the outer equation in (t,x) at level $i-2$.)

A_2^{i-2} contains terms at the level $i-2$ or lower, i.e., already known quantities, and all of them contain at least one derivative with respect to z, so they vanish as $|z| \to \infty$. A_1^{i-2} vanish as $|z| \to \infty$ because of the outer expansion equations.

As in ref. 10, we need in order to have a bounded solution at the *next* level: $\int_{\mathbb{R}} (A^i(t,x,z) + A_1^{i-1}(t,x,z) + A_2^{i-1}(t,x,z))\, dz = 0$.

Let $[u] := u(t,x,+\infty) - u(t,x,-\infty)$, then we get the following condition for each (t,x) in a δ-neighborhood of the free boundary:

$$\begin{aligned} 0 = {} & 2m_\beta \partial_t d^i + (\partial_t d^0 - \Delta u^0)[u_i] + (\partial_t d^i - \Delta u_i)[u^0] \\ & - 2(\nabla d^i [\nabla u_0] + \nabla d^0 [\nabla u_i]) + (d^0 l_i + l_0 d^i) + h_i. \end{aligned} \tag{2.10}$$

The corrected expansion equation for the order parameter has the following form:

Replace in (2.8) the terms B_p^i and B_T^i by $\hat{B}_p^i$ and $\hat{B}_T^i$: $\hat{B}_p^i := B_p^i + N_1(d^i g^0 + g^i d^0)$, $\hat{B}_T^i := B_T^i + (1 - \bar{m}(z)^2 - \bar{m}'(z)\, N_1)(d^i g^0 + g^i d^0)$, $N_1 = \int_{\mathbb{R}} \bar{m}'(z)(1-\bar{m}(z)^2)^{-1}\, \eta'(z)\, dz$. Then add to B^{i-1} the terms depending on $\{d^j g^{i-1-j}\}_{j=0}^{i-1}$. The resulting solvability condition (right hand side orthogonal to $\bar{m}'$) $B_p^i + \langle \bar{m}', B^{i-1} \rangle_{L^2(\mu(dz))} = 0$ is similar as in ref. 10:

$$\begin{aligned} 0 = {} & \partial_t d^i(t,x) - \theta \Delta d^i(t,x) - \overline{u_i}(t,x) \\ & + N_1(d^i(t,x)\, g^0(t,x) + g^i(t,x)\, d^0(t,x)) + b_{i-1}(t,x), \end{aligned} \tag{2.11}$$

b_{i-1} depends only on known quantities.

From regularity for $L_{\bar{m}}^{(1)}(m_i) = R^{i-1}$ in weighted spaces (see Appendix) we get exponential convergence of m_i and its derivatives as $r \to \pm\infty$.

2.3. Existence of the Asymptotic Expansions

We remark that short-time existence and regularity for Eqs. (1.1) and (1.2) is straightforward by a fixed-point argument: Rewrite both equations as integral equations using the variation of constants formula.

The existence on $[0, T_{\max} - \delta)$ will be a by-product of the convergence proof: As long as the solution exists, it stays close to the constructed approximate solution, thus we can iterate the short-time existence.

2.3.1. The First Order

We briefly sketch the strategy for solving the corrected expansion equations, as it is exactly the same as in ref. 10. We start with the first order equation, which is slightly different because it leads to the limit free boundary problem.

We immediately get $u_0(t,x,z) = \hat{u}(t,x) + h_0(t,x)\, d^0(t,x)\, \eta(z)$. First consider $(t,x) \in \Sigma^0(t)$, i.e., $d^0(t,x) = 0$. Then (2.11) gives $-2Nm_\beta \hat{u}(t,x) + V_{\Sigma^0} - \theta \kappa_{\Sigma^0} = 0$, which is one of the equations fulfilled by the limit Stefan problem. So $\hat{u}$ and thus $u_0(t, x, z)$ is fixed on the free boundary by the solution of the limit problem. By the matching conditions, this defines the boundary values for the outer expansion, and thus $u_0^{\pm}(t, x)$ is fixed. The matching conditions for general (t, x) determine h_0, because

$$u_0^+(t, x) = \hat{u}(t, x) + h_0(t, x)\, d^0(t, x), \qquad u_0^-(t, x) = \hat{u}(t, x),$$

so $h_0(t, x) = d^0(t, x)^{-1}\,(u_0^+(t, x) - u_0^-(t, x))$. Thus we get $h_0(t, x) = [\nabla u_0]$ on $\Sigma^0(t)$.

Now consider (2.10) for $(t, x) \in \Sigma^0(t)$. As $[u_0] = 0$, we are left with $2m_\beta V|_{\Sigma^0} = [\nabla u_0]$, which is the Stefan condition at the interface. Finally we define l_0 by requiring that (2.10) is solved also away from the interface.

2.3.2. The Higher Orders

As ref. 10, we end up with a linear system of PDEs:

$$\begin{aligned} 2m_\beta \partial_t d^j &= [\nabla u_j]_\Sigma + e_1^{j-1} d^j + e_2^{j-1} && \text{on} \quad \Sigma^0 \\ \partial_t u_j^{\pm} &= \Delta u_j^{\pm} + e_3^{j-1} && \text{in} \quad \Omega^{\pm} \\ u_j^{\pm} &= \partial_t d^j - \theta \Delta d^j + e_{5\pm}^{j-1} d^j + e_1^{j-1} && \text{on} \quad \Sigma^0 \\ \nabla d^0\, \nabla d^j &= e_4^{j-1} && \text{in} \quad U_\delta(\Sigma) \end{aligned}$$

with initial conditions and periodic boundary conditions. The e^{j-1} depend only on already computed quantities. Existence can be shown by Banach fixed point arguments. (For details see refs. 10 and 13.)

2.4. Construction of the Approximate Solution

We have to use cut-off functions and the *exponentially fast* convergence given by the matching conditions to construct out of the inner and outer expansions an approximate solution to the equation.

Take the first K terms of the expansions for d^λ, u^λ, m^λ, $u^{\lambda,\pm}$, and $m^{\lambda,\pm}$, set for the inner expansions $z = \lambda d^K$, and denote the resulting functions by m^K, etc.

Let $u_O^K := 1_{\Omega^+(t)} u^{+,K} + 1_{\Omega^-(t)} u^{-,K}$, and define m_O^K in the same way. Take a smooth cut-off function $\zeta \in C_0^\infty(-1, 1)$, $\zeta \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$. Now define

$$\begin{aligned} u_a &= u_O^K && \text{in} \quad \Omega \backslash U_\delta(\Sigma) \\ u_a &= u_I^K \zeta\left(\frac{d^0}{\delta}\right) + \left(1 - \zeta\left(\frac{d^0}{\delta}\right)\right) u_O^K && \text{in} \quad U_\delta(\Sigma) \backslash U_{\frac{\delta}{2}}(\Sigma) \\ u_a &= u_I^K && \text{in} \quad U_{\frac{\delta}{2}}(\Sigma), \end{aligned}$$

and define m_a in the same way.

The only region where problems could arise from is the coexistence region of outer and inner expansion. Due to the exponential matching conditions we get that if the asymptotic solutions solved the equations formally up to order K, then u_a, m_a are solutions up to a right hand side of order λ^{K-1} in L^∞.

3. CONVERGENCE PROOF

We will show Theorem 1.1 (2) by performing step 2 and step 3 as outlined in the introduction, Section 1.3.

3.1. The Linearization

In this subsection we will linearize the equation (1.1) around the approximate solution. In the sequel we will denote by Ψ the variations of the order parameter m and by v the variations of the temperature field u.

Consider the approximate solution constructed above up to and including the first correction to the instanton, i.e., for $K=2$. As K will remain fixed, we denote the approximate solution for $K=2$ by m_A, u_A. Σ^λ, the zero level set of $d^\lambda(t,x)$, is a sufficiently regular compact hypersurface. Let

$$h_A(t,x) := \beta(J^\lambda * m_A + \lambda u_A) \quad \text{and} \quad a(t,x) := \beta(1-\tanh(h_A(t,x))^2). \tag{3.1}$$

The linearized system is:

$$\begin{aligned} \partial_t \Psi &= -\lambda^{-2}\Psi + \lambda^{-2}\beta(1-\tanh(h_A(t,x))^2\,(J^\lambda * \Psi + \lambda v). \\ \partial_t v &= \Delta v - \partial_t \Psi. \end{aligned} \tag{3.2}$$

In order to symmetrize the nonlocal (convolution) part of the operator we consider the evolution of $\Psi_a = \Psi(t,x)\,a(t,x)^{-\frac{1}{2}}$.

Clearly $C^{-1}\,\|\Psi\|_{L^2(\mathrm{d}x)} \leqslant \|\Psi_a\|_{L^2(\mathrm{d}x)} = \|\Psi\|_{L^2(a^{-1}(x)\,\mathrm{d}x)} \leqslant C\,\|\Psi\|_{L^2(\mathrm{d}x)}$ for some $C>0$. We are interested in $\partial_t\,\|\Psi\|^2_{L^2}$, because that is what we have to control in order to make the method of the convergence proof work:

We have to show linear growth with coefficients uniform in λ. Let Ψ solve (3.2). $\frac{1}{2}\,\partial_t(\Psi_a)^2 = (\partial_t\Psi)\,a^{-1}\Psi - \frac{1}{2}\,(\partial_t a)\,a^{-2}\Psi^2$, so

$$\frac{\lambda^2}{2}\frac{d}{dt}\,\|\Psi(t,x)\|^2_{L^2(a^{-1}(x)\,\mathrm{d}x)} = -Q^\lambda(\Psi) + \lambda v\psi,$$

$$Q^\lambda(\Psi) = \tfrac{1}{2}\iint J^\lambda(x-y)(\Psi(x)-\Psi(y))^2\,\mathrm{d}x\,\mathrm{d}y + \int f^\lambda(t,x)\,\Psi(x)^2\,\mathrm{d}x, \tag{3.3}$$

$$f^\lambda(t,x) = \frac{1}{a(x)} - 1 + \lambda^2\,\frac{\partial_t a}{2a^2}.$$

f^λ depends on the approximate free boundary through $d^\lambda(t,x)$ and the approximate normal velocity $\partial_t d^\lambda(t,x)$, and it depends on the approximate temperature u_A and the order parameter m_A. We write, omitting time dependence, $r^\lambda(x) := \mathrm{dist}(x, \Sigma^\lambda(t)) = d_A^\lambda(t,x)$ and find by straightforward but tedious computations:

Lemma 3.1. There are $0 < c_* \leqslant C^*$ independent of λ such that $f^\lambda(x) > c_*$ on $|d^\lambda(x)| > C^*\lambda$. Moreover there is a $\delta > 0$ independent of λ such that the following holds for $|d^\lambda(x,t)| < \delta$: There is (uniformly in t) a λ_0 and a C independent of λ such that for any $\lambda < \lambda_0$:

$$f^\lambda(x) = f_0\left(\frac{r^\lambda(x)}{\lambda}\right) + \lambda f_1\left(x, \frac{r^\lambda(x)}{\lambda}\right) + f_R^\lambda(x),$$

$$f_0\left(\frac{r^\lambda(x)}{\lambda}\right) := \left(\beta\left(1 - \bar{m}\left(\frac{r^\lambda(x)}{\lambda}\right)^2\right)\right)^{-1} - 1, \tag{3.4}$$

$$f_1\left(x, \frac{r^\lambda(x)}{\lambda}\right) := \frac{2\bar{m}\left(\frac{r^\lambda(x)}{\lambda}\right)\left[m_0\left(x, \frac{r(x)}{\lambda}\right) + \frac{\partial_t d^\lambda(x)}{2}\bar{m}'\left(\frac{r^\lambda(x)}{\lambda}\right)\right]}{\beta\left(1 - \bar{m}\left(\frac{r^\lambda(x)}{\lambda}\right)^2\right)^2}, \tag{3.5}$$

$$|f_R^\lambda(x)| \leqslant \lambda^2 C(1 + \lambda^{-1}|r^\lambda(x)|). \tag{3.6}$$

Note that for some constant $0 < C(J,\beta)$ we have $f_0(r) < 0$ for $|r| < C(J,\beta)$ (concave part) and $f_0(r) > 0$ for $|r| > C(J,\beta)$. (Convex part)

We briefly comment on the appearance of Σ^λ, r^λ instead of the limit surface Σ^0. First we remark that we could use the first order correction Σ^1, but not Σ^0. The reason is that $\|\bar{m}(\lambda^{-1}r^0(x)) - \bar{m}(\lambda^{-1}r^\lambda(x))\|_\infty = O(1)$, which would spoil the estimates. The surface Σ^1 (which we could take instead of Σ^λ) is a shift of order λ in the normal direction of the surface Σ^0, which is in the tangential direction C^K, K large enough by construction. So all estimates involving derivatives of the surface like curvature etc. are uniform in λ. For this reason we will drop the dependence on λ of the distance, velocity and other geometric quantities in the rest of the proof, moreover we write $r(x)$ instead of $r^\lambda(x)$ for the distance from the zero level set.

3.2. Spectral Analysis

The main result of this subsection is

Lemma 3.2. Spectral estimate: There is a constant $C > 0$ not depending on λ such that

$$\inf_{\|\Psi(x)\|_{L^2(d^n x)} = 1} Q^\lambda(\Psi) \geqslant -C\lambda^2$$

for $\lambda < \lambda_0$, where λ_0 depends only on the approximate solution and on a priori known constants.

We call it *spectral estimate*, as it is a bound on the spectrum of the symmetric linear operator which is associated with the quadratic form $Q(\Psi)$. Of course this is an estimate for Ψ only, but the coupling is weak, and the v-equation linear and parabolic (regularizing). So the Ψ-estimate turns out to be the crucial step.

The structure of the proof is as follows: First we provide a very rough estimate, which tells something about the structure of the radial average of Ψ^2. It can be improved, but the improvements do not seem to simplify anything later.

Then we will compute a *formal* asymptotic expansion of the "order λ^2" eigenfunction. This together with a "Perron–Frobenius" trick will give the lemma. It is crucial that the highest order part of the unscaled *one-dimensional* version of Q^λ has a *minimum* zero which is attained by a strictly positive function, $\bar{m}'$, and that in one dimension there is a "spectral gap" independent of λ.

The part $\iint J^\lambda(x-y)(\Psi(x)-\Psi(y))^2 \,\mathrm{d}x\,\mathrm{d}y$ will be called the *interaction part*, $\int f^\lambda \Psi^2 \,\mathrm{d}x$ the *local part*.

In a δ-neighborhood of a smooth interface we can use coordinates $\phi = \mathrm{Pr}_\Sigma(x)$ and $r = \mathrm{dist}(x, \Sigma)$. Let $\Sigma_r := \{x \in \mathbb{R}^n \mid \mathrm{dist}(x, \Sigma) = r\}$ for $|r| < \delta$. For any $\Psi \in L^2$ and any $|r| < \delta$, $x \in U_\delta(\Sigma)$, define

$$\bar{\Psi}(r) := \sqrt{\int_{\Sigma(r)} \Psi^2(x)\, \mathscr{H}^{n-1}|_{\Sigma_r}(\mathrm{d}x)}, \qquad \hat{J}^\lambda(x, r) := \int_{\Sigma_r} J^\lambda(x, y)\, \mathscr{H}^{n-1}|_{\Sigma_r}(\mathrm{d}y).$$

Lemma 3.3. There is a C independent of λ s. t.

$$|\hat{J}^\lambda(x, r) - \bar{J}^\lambda(r(x)-r)| \leqslant \lambda[C\lambda^{-1} 1_{|r(x)-r| < \lambda C}]$$

on $U_\delta(\Sigma) \times U_\delta(\Sigma)$. (The expression in brackets is a scaled 1-d kernel)

Sketch of the proof: We write the parallel surface Σ_r as a graph of the function f over the tangent plane T_{p_r} in a point $p_r \in \Sigma_r$ such that $x - p_r$ is normal to Σ_r and Σ. Expand by Taylor's formula f to the second and ∇f to the first order around p_r. Then the estimate follows from the Lipschitz-continuity and rotation invariance of J.

Lemma 3.4. Take $\lambda \ll 1$. Assume that $\|\Psi\|_2 = 1$, $Q^\lambda(\Psi) \leqslant C\lambda^{\frac{1}{2}}$. Let $\Psi_\delta := \Psi 1_{r \leqslant \delta}$. Then $\|(\overline{\Psi_\delta})^\perp\|^2_{L^2(\mathbb{R})} \leqslant \lambda^{\frac{1}{2}}$. $f^\perp$ denotes the part of $f \in L^2(\mathbb{R})$ orthogonal to $\bar{m}'$ in $L^2(\mu)$, see Appendix.

(Ψ as in the lemma exist, e.g., $\lambda^{-\frac{1}{2}} \bar{m}'(r(x)\,\lambda^{-1})$.)

Proof. First we introduce a symmetric cut-off function $\xi^\lambda\colon \mathbb{R} \to [0,1]$ s.t. $|\xi'(r)| \leqslant C(\delta)$ and $\xi(r) = 1$ for $|r| \leqslant \delta/2$, $\xi(x) = 0$ for $|r| \geqslant \frac{3}{4}\delta$.

Now take a function $\tilde{\Psi}$ as in the statement of the lemma. As the local part of the functional is convex for $|\mathrm{dist}(x, \Sigma)| \in [\delta/2, \infty)$, we get from the assumption on the derivative of ξ and standard convolution estimates

$$Q^\lambda(\tilde{\Psi}) + \lambda(\|\xi'\|_\infty)\, \|\Psi 1_{\mathrm{supp}(J^\lambda * \xi)}\|_2^2 \geqslant Q^\lambda(\xi\tilde{\Psi}), \text{ so}$$

$$Q^\lambda(\tilde{\Psi}) \geqslant Q^\lambda(\xi\tilde{\Psi}) - C\lambda\, \|\tilde{\Psi}\|_2^2. \tag{3.7}$$

To simplify notation, we will from now on write $\Psi := \tilde{\Psi}\xi$.

Both the highest order of the local part of Q^λ and the L^2-Norm are unchanged under the rearrangement $\Psi \to \bar{\Psi}$. Now we consider the nonlocal part of $Q^\lambda(\Psi)$. By the coarea formula

$$\tfrac{1}{2}\iint_{\mathbb{R}^n \times \mathbb{R}^n} J^\lambda(x-y)(\Psi(x) - \Psi(y))^2 \,\mathrm{d}x\,\mathrm{d}y = \int \bar{\Psi}^2(r)\,\mathrm{d}r$$

$$-\iint \underbrace{\iint [\sqrt{J^\lambda(x-y)}\,\Psi(x)][\sqrt{J^\lambda(y-x)}\,\Psi(y)]\, \mathscr{H}^{n-1}_{\Sigma_r} \times \mathscr{H}^{n-1}_{\Sigma_{r'}}(\mathrm{d}x, \mathrm{d}y)}_{I^\lambda[\Psi](r,r')} \,\mathrm{d}r\,\mathrm{d}r'.$$

We wish to replace the interaction part by a 1-d expression depending only on $\bar{\Psi}$ and $\bar{J}^\lambda$. Apply the Cauchy-Schwarz inequality to $I^\lambda[\Psi](r,r')$ and integrate over Σ to get

$$I^\lambda[\Psi](r,r') \leqslant \sqrt{\int \hat{J}^\lambda(x,r')\,\Psi^2(x)\, \mathscr{H}^{n-1}_{\Sigma_r}(\mathrm{d}x)}\ \sqrt{\int \hat{J}^\lambda(x,r)\,\Psi^2(x)\, \mathscr{H}^{n-1}_{\Sigma_{r'}}(\mathrm{d}x)}.$$

Replace $\hat{J}(x,r)$ by $\bar{J}(r,r')$ with Lemma 3.3, using the Hölder-continuity of the square root. Note that $\bar{J}(r,r')$ is symmetric in its variables. We get

$$I^\lambda[\Psi](r,r') \leqslant \bar{J}^\lambda(r-r')\,\bar{\Psi}(r)\,\bar{\Psi}(r') + C\lambda^{-\frac{1}{2}}\, 1_{|r-r'| \leqslant \lambda}(r,r')\,\bar{\Psi}(r)\,\bar{\Psi}(r'),$$

so finally by convolution estimates

$$Q^\lambda(\Psi) \geqslant \bar{Q}^\lambda(\bar{\Psi}) - \lambda^{\frac{1}{2}} C\, \|\Psi\|_2^2,$$

where $\bar{Q}$ is the 1-d functional on $L^2(\mathbb{R})$ with convolution kernel $\bar{J}$, as studied for example in ref. 19. The unscaled 1-d functional has zero as minimum, attained by $\bar{m}'(r)$, and is bounded away from zero on functions $f \in L^2(\mathbb{R})$, $\|f\|_2 = 1$, which are orthogonal to $\bar{m}'$. (Spectral gap.) So we get $Q^\lambda(\tilde{\Psi}) + \lambda^{\frac{1}{2}}\|\Psi\|_2^2 \geqslant Q^\lambda(\Psi) \geqslant c\, \|\bar{\Psi}^\perp\|_2^2 - C\lambda^{\frac{1}{2}}\, \|\Psi\|_2^2$, which implies $\|\bar{\Psi}^\perp\|_2^2 \leqslant C\lambda^{\frac{1}{2}}$. (Remember $Q^\lambda(\tilde{\Psi}) \leqslant \lambda^{\frac{1}{2}}$.) ∎

Lemma 3.5 (Approximate Eigenfunction). Let f_0 and f_1 be the zeroth and first order of the local part as in (3.4) and (3.5), and let

$$L^\lambda(\Psi)(x) := (J^\lambda * \Psi)(x) - \left\{ \left[f_0\left(\frac{r(x)}{\lambda}\right) + 1 \right] + \lambda f_1\left(\mathrm{Pr}_\Sigma(x), \frac{r(x)}{\lambda} \right) \right\} \Psi(x).$$

Then there is a smooth function $P(s, r)\colon \Sigma \times \mathbb{R} \to \mathbb{R}$ such that

$$\Phi(x) := \bar{m}'(\lambda^{-1} r(x)) + \lambda P(\mathrm{Pr}_\Sigma(x), \lambda^{-1} r(x))$$

solves on $\{|\mathrm{dist}(x, \Sigma)| < \delta/2\}$

$$L^\lambda(\Phi)(x) = \lambda^2 H(x),$$

where $|H(x)| \leqslant C(1 + |\lambda^{-1} r(x)|)\, \bar{m}'(\lambda^{-1} r(x))$, and $|P(s, r)| + |P'(s, r)| \leqslant C(1 + |r|)\, \bar{m}'(r)$.

Remark 3.6. By computing one more step of the expansion, we could construct a function $e_\lambda^2 := \bar{m}' + \lambda P + \lambda^2 P_2$, which is a formal eigenfunction, i.e., $L^\lambda(e_\lambda^2) = c\lambda^2 e_\lambda^2 + O(\lambda^3)$.

Proof. We put the ansatz in the equation, expand the convolution as in Section 2 to replace it by a one-dimensional convolution plus error terms and find by straightforward but lengthy computations that P must satisfy

$$\int_{\mathbb{R}} \bar{J}(r - r')\, P(s, r')\, dr' - \frac{P(s, r)}{\beta(1 - \bar{m}^2(r)^2)} = -R(\kappa(s), r) - \bar{m}'(r)\, A(r),$$

$$A(r) := \frac{2\bar{m}(r)[m_0(s, r) + \frac{1}{2} V(s)\, \bar{m}'(r)]}{\beta(1 - \bar{m}^2(r)^2)^2},$$

$$R(\kappa(s), r) := \int \bar{J}_{\tan}(|r - r'|)\, \bar{m}''(r')\, \kappa(s)\, |\mathbf{y}|^2\, \mathrm{d}r'\, \mathcal{L}^{n-1}(\mathrm{d}\mathbf{y}), \tag{3.8}$$

$$\bar{J}_{\tan}(r - r') := \int J\left(\sqrt{|r - r'|^2 + |\mathbf{y}|^2} \right) |\mathbf{y}|^2\, \mathcal{L}^{n-1}(\mathrm{d}\mathbf{y}),$$

where s is treated as parameter, and the *tan* in $\bar{J}_{\tan}$ stands for "tangential average." We denote by $\kappa(s)$ the mean curvature in the point $s \in \Sigma$, $V(s)$ the normal velocity and let $\mathbf{y} = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}$.

The ideas is as always that P cancels the error from inserting the first order part in the equation. Due to the decay properties of P, the residual H is actually of the required form.

P exists according to Section 4.1 because the right-hand side of (3.8) is bounded by $C\bar{m}'(r)$ for some C independent of λ and is orthogonal to $\bar{m}'$. The boundedness follows from basic properties of the instanton and its derivatives, (see ref. 7, (2.8))): $|\bar{m}''(r)(\bar{m}'(r))^{-1}| \leqslant C$, $|\bar{m}'''(r)(\bar{m}'(r))^{-1}| \leqslant C$.

In order to show that the right hand side of (3.8) is orthogonal to $\bar{m}'$, we use symmetry arguments: $\bar{m}'$ is symmetric. From the symmetry of the kernel, the symmetry properties of the instanton and its derivatives and the equation defining m_0 near the interface we derive that the right hand side is antisymmetric and thus orthogonal to $\bar{m}'$. ▮

Now we are able to give a proof for Lemma 3.2:

Proof. Let $h_0(r) := (\beta(1-\bar{m}^2(r)))^{-1} = f^0(r)+1$ and let $f_1(x, r)$ be as before.

We need to introduce a cut-off which is 0 where $\lambda\,|P| > \bar{m}'$, and which is close to 1 in the region where $f_0 = h_0 - 1 < 0$. Unfortunately we cannot take just any cut-off. We need a cut-off ζ for which $|\zeta_\lambda'| \leqslant C\lambda^{-1}(1-\zeta^2)$. So we choose a cut-off ζ_λ in the following way:

$$\zeta_\lambda(r) = \begin{cases} \tanh\left(K\,|\log(\lambda)| - \alpha\dfrac{|r|}{\lambda}\right) & \text{on} \quad \alpha\,|r| \leqslant K\lambda(|\log(\lambda)|-1), \\ 0 & \text{on} \quad \alpha\,|r| \geqslant K\lambda(|\log(\lambda)|), \end{cases}$$

and $\zeta_\lambda(r(x))$ is smooth on $\mathbb{R}^n$. (The function defined above can easily be extended to a smooth function with $|\zeta_\lambda'(r)| \leqslant \alpha\lambda^{-1}$.) The constants c_1, α, K will be determined later.

We write $Q(\Psi) = l - I = \int (h_0 + \lambda f_1)\,\Psi^2 - \iint J^\lambda(x-y)\,\Psi(x)\,\Psi(y)\,dx\,dy$ and split the interaction part in two: $I = I_1 + I_2$,

$$I_1 = \iint J^\lambda(x-y)\,\Psi(x)\,\zeta_\lambda(x)\,\Psi(y)\,\zeta_\lambda(y)\,dx\,dy$$

$$I_2 = \iint J^\lambda(x-y)(1-\zeta_\lambda(x)\,\zeta_\lambda(y))\,\Psi(x)\,\Psi(y)\,dx\,dy.$$

We split the local part l with the cut-off:

$$l_1 = \int_{\mathbb{R}^n} [(\zeta_\lambda\Psi)^2\,h_0 + \lambda f_1(\zeta_\lambda\Psi)^2], \qquad l_2 = \int_{\mathbb{R}^n} (1-\zeta_\lambda^2)(h_0\Psi^2 + \lambda f_1\Psi^2).$$

Hence $Q(\Psi) = l_1 + l_2 - (I_1 + I_2)$. First consider $Q_1 = l_1 - I_1$. For I_1

$$\iint J^\lambda(x-y)(\Psi\zeta_\lambda)(x)(\Psi\zeta_\lambda)(y)\,dx\,dy = \iint J^\lambda(x-y)(\Phi S)(x)(\Phi S)(y)\,dx\,dy.$$

$S(x) = \Psi(x)\,\zeta_\lambda(x)(\Phi(x))^{-1}$, which is well defined, because $\Phi > 0$ on $\operatorname{supp}(\zeta_\lambda)$. Now use $-ab = \frac{1}{2}(a-b)^2 - \frac{1}{2}(a^2+b^2)$ with $ab = S(x)\,S(y)$ and the symmetry of J to derive

$$-\iint J^\lambda(x-y)\,\Psi(x)\,\zeta_\lambda(x)\,\Psi(y)\,\zeta_\lambda(y)\,\mathrm{d}x\,\mathrm{d}y = -K_1 + K_2,$$

$$K_1 := \iint J^\lambda(x-y)\,\Phi(x)(\Phi(y)\,S^2(y))\,\mathrm{d}x\,\mathrm{d}y,$$

$$K_2 := \tfrac{1}{2}\iint J^\lambda(x-y)\,\Phi(x)\,\Phi(y)(S(x)-S(y))^2\,\mathrm{d}x\,\mathrm{d}y.$$

$K_2 \geqslant 0$ on $\operatorname{supp}(\zeta)$. Note that $S(x) = 0$ if $\zeta_\lambda(x) = 0$.

In K_1 we integrate over x and replace $J * \Phi$ by Lemma 3.5, such that the result is l_1 up to a small error. We get

$$\begin{aligned} -K_1 &= -\int_{\mathbb{R}^n} (h_0(\lambda^{-1}r(x)) + \lambda f_1(x, \lambda^{-1}r(x)))\,\Phi(x)\,S^2(x)\,\Phi(x)\,\mathrm{d}x + \lambda^2 \bar{H} \\ &= -l_1 + \lambda^2 \bar{H}, \qquad |\bar{H}| \leqslant \int C(1+\lambda^{-1}\,|r(x)|)(\zeta_\lambda(x)\,\Psi(x))^2. \end{aligned}$$

For the last estimate we used that $\nabla_x m_0$ is uniformly bounded in $U_\delta(\Sigma)$ by construction. Thus $Q(\Psi) \geqslant l_2 - I_2 - \lambda^2\,|\bar{H}|$.

We estimate $-I_2$ after some algebraic manipulations as

$$\begin{aligned} -I_2 &\geqslant +\tfrac{1}{2}\iint (1-\zeta_\lambda(x)\,\zeta_\lambda(y))(\Psi(x)-\Psi(y))^2 - \int (1-\zeta_\lambda(x)^2)\,\Psi(x)^2 \\ &\quad - \iint J^\lambda(x-y)\left[\,|x-y|\int_0^1 |\nabla\zeta_\lambda((1-s)\,x+y)|\,ds\,\right]\zeta_\lambda(x)\,\Psi^2(x) \\ &\geqslant -\int (1-\zeta_\lambda(x)^2)\,\Psi(x)^2 - C\int \alpha(1-\zeta_\lambda^2(x))\,\zeta_\lambda(x)\,\Psi^2(x), \end{aligned}$$

where we used that ζ_λ is a hyperbolic tangent, i.e., $\zeta_\lambda' = \alpha\lambda^{-1}(1-\zeta_\lambda^2)$. Making use of the exponential decay of the cut-off, we derive

$$l_2 - I_2 - \lambda^2\,|\bar{H}| \geqslant \int (h_0 - 1 - \lambda\,\|f_1\|_\infty - C\alpha)(1-\zeta_\lambda(x)^2)\,\Psi(x)^2 - \lambda^2\,|\bar{H}|.$$

There are α_0 and $\tilde{C}^*$ independent of $\lambda < \lambda_0$, such that

$$f_0(r) = h_0(r) - 1 = \frac{1 - \beta(1 - \bar{m}(r)^2)}{\beta(1 - \bar{m}(r)^2)} > C\alpha + \lambda \, \|f_1\|_\infty$$

for $|r| > \tilde{C}^*$, $\alpha < \alpha_0$, so $(h_0 - 1 - \lambda \, \|f_1\| - C\alpha) \geqslant 0$ for $|r(x)| > \lambda \tilde{C}^*$.

As the hyperbolic tangent converges exponentially fast to ± 1, we can choose K such that $1 - \zeta_\lambda^2(r) \leqslant \lambda^2$ on $|r(x)| < \lambda \tilde{C}^*$, so we have $l_2 - I_2 \geqslant - C\lambda^2 \, \|\Psi 1_{|r(x)| \leqslant \lambda \tilde{C}^*}\|_2^2 - \lambda^2 \, |\bar{H}|$, and we need only an estimate for $|\bar{H}|$. We use Lemma 3.4 for a bound not depending on λ.

$$\begin{aligned}
|\bar{H}| &\leqslant \int_{C^* \lambda \leqslant r(x) \leqslant \lambda \, |\log(\lambda)|} C(1 + \lambda^{-1} \, |r(x)|) \, \Psi(x)^2 \, \mathrm{d}x \\
&= \int_{C^* \lambda}^{\lambda \, |\log(\lambda)|} C \left(1 + \frac{|r|}{\lambda}\right) \bar{\Psi}(x)^2 \, \mathrm{d}r \\
&\leqslant \int_{C^* \lambda}^{\lambda \, |\log(\lambda)|} C \left(1 + \frac{|r|}{\lambda}\right) \left[\lambda^{-1} \bar{m}' \left(\frac{r}{\lambda}\right)^2 + (\bar{\Psi}(x)^{\perp})^2\right] \mathrm{d}r \\
&\leqslant C + C \, |\log(\lambda)| \, \lambda^{\frac{1}{2}} \leqslant \hat{C}. \qquad \blacksquare
\end{aligned}$$

3.3. The Gronwall Estimate

Let m, u be the solutions of (1.1), (1.2), and u_a, m_a the approximate solutions from Section 2. We define

$$\Psi(t, x) := m - m_a(t, x), \quad v := u - u_a, \quad w(t, x) := \int_0^t (u(s, x) - u_a(s, x)) \, \mathrm{d}s.$$

(See also ref. 10). We assume that $\|\Psi(0)\|_{L^\infty} + \|w(0)\|_{L^\infty} \leqslant C\lambda^p$, $p > n + 6$, and that (1.1) is solved by m_a up to a right hand side r_a^λ with $\|r_a^\lambda\|_2^2 \leqslant \lambda^p$, and that the energy equation is solved exactly by (m_a, u_a). Let $T^* := \min(T_{\max} - \delta, \inf\{t \colon \|\Psi(t)\|_{L^2} > \lambda^{\frac{n+4}{2}}\}, \inf\{t \colon \|u\|_{L^\infty} \geqslant \|u_a\|_{L^\infty} + 1\})$.

Ψ and w are smooth in space and time on $[0, T^*)$ and $|\Psi| \leqslant 2$.

As in Section 3.1, we write $a^{-1} \, \mathrm{d}x$ for the measure $a^{-1}(t, x) \, \mathrm{d}x$, where $a(t, x)$ is as in (3.1). Let $\|\cdot\|_a$ denote the L^2 norm with respect to $a^{-1}(t, x) \, \mathrm{d}x$. Then we get from (3.2) and (3.3) and taking into account the linearization error on $[0, T^*]$:

$$\tfrac{1}{2} \partial_t \, \|\Psi\|_a^2 = - \lambda^{-2} Q^\lambda(\Psi) + \int \{\lambda^{-1}(\partial_t w) \, \Psi + \lambda^{-2} R(x, t, \Psi, \lambda \partial_t w) \, \Psi\} \, \mathrm{d}x \tag{3.9}$$

$$\partial_t w = \Delta w - \Psi. \tag{3.10}$$

As the hyperbolic tangent has bounded second derivative, we know

$$|R(x, t, J^{*}\Psi, \lambda\partial_t w)| \leqslant C[(J * \Psi)^2 + r_a^\lambda + \lambda^2(\partial_t w)^2].$$

Further $(\partial_t w)^2 \, |\Psi| \leqslant 2(\partial_t w)^2$. Put $N_\lambda(\Psi) := C\lambda^{-2}[\int (J^\lambda * \Psi)^2 \, |\Psi| \, \mathrm{d}x + Cr_a^\lambda]$.

Now substitute the second equation into the first and use the spectral estimate for Q^λ, Lemma 3.2, which gives

$$\tfrac{1}{2}\,\partial_t \, \|\Psi\|_a^2 \leqslant C \int \{\Psi^2 + \lambda^{-1}(\Delta w - \Psi)\,\Psi + c(\Delta w - \Psi)^2\} \, \mathrm{d}x + N_\lambda(\Psi).$$

In order to get rid of the term $\Delta w \Psi$, multiply (3.10) by $-\lambda^{-1}\,\Delta w$, then integrate by parts in space on the left hand side (w is a difference and thus has zero boundary conditions), exchange time and space derivative (u and u_A are regular locally in time,) and use the result to get (see also ref. 10):

$$\frac{1}{2}\,\partial_t \, \|\Psi\|_a^2 + \frac{1}{2\lambda}\,\partial_t \int |\nabla w|^2 \leqslant C\,\|\Psi\|_2^2 + \int \lambda^{-1}(-(\Delta w)^2 + 2\Delta w \Psi - \Psi^2) \, \mathrm{d}x$$
$$+ C \int (\Delta w - \Psi)^2 \, \mathrm{d}x + CN_\lambda(\Psi).$$

So for $\lambda < \lambda_0(c, C)$:

$$\tfrac{1}{2}\,\partial_t \, \|\Psi\|_a^2 + C\lambda^{-1}\left(\partial_t \int |\nabla w|^2 + \|v\|_{L^2(\Omega)}^2\right) \leqslant C\,\|\Psi\|_2^2 + N_\lambda(\Psi).$$

Integrate over time and use the equivalence of $L^2(a^{-1}\,\mathrm{d}x)$ and $L^2(\mathrm{d}x)$:

$$\|\Psi(t)\|_2^2 + C\lambda^{-1}(\|\nabla w(t)\|_2^2 + \|v\|_{L^2([0,T^*]\times\Omega)}^2)$$
$$\leqslant C\lambda^{p-2} + C'' \int_0^t \|\Psi(s)\|_2^2 \, \mathrm{d}s + C'' \int_0^t \int_{\mathbb{R}^n} \lambda^{-2}(J^{\lambda *}\Psi)^2 \, |\Psi| \, \mathrm{d}x \, \mathrm{d}s.$$

It remains to treat the term which is formally of third order in Ψ. As $\|J^\lambda * \Psi\|_{L^\infty} \leqslant C\lambda^{-n/2}\,\|\Psi\|_{L^2}$, we arrive at

$$\int (J^\lambda * \Psi)^2\,(x)\,\Psi(x) \leqslant (\|J^\lambda * \Psi\|_{L^2})(\|(J^\lambda * \Psi)\,\Psi\|_{L^2})$$
$$\leqslant C(\lambda^{-n/2}\,\|\Psi\|_{L^2})(\|\Psi\|_{L^2}^2).$$

Hence until T^*, we can apply Gronwall's inequality and get

$$\begin{aligned} &\text{(i)} \quad \|\Psi(t)\|_2^2 \leqslant C(T)\,\lambda^{p-2}, \\ &\text{(ii)} \quad \|\nabla w(t)\|_2^2 \leqslant C(T)\,\lambda^{p-1}, \\ &\text{(iii)} \quad \int_0^{T^*} \int v^2(t,x)\,\mathrm{d}x\,\mathrm{d}t \leqslant C(T)\,\lambda^{p-1}. \end{aligned} \tag{3.11}$$

Now we wish to derive convergence in better norms.

First we get a $L^\infty([0,T], L^2(\Omega))$-bound for $v = u - u_a$. Remember $\lambda^2 \partial_t \Psi = -\Psi + \tanh(\beta J^\lambda * m_a + \beta\lambda u_a) - \tanh(\beta J^\lambda * m + \beta\lambda u)$. By testing the equation fulfilled by v, i.e., (1.2), with v we get

$$\|v(t)\|_2^2 \leqslant \|v(0)\|_2^2 + \int_0^t \int_\Omega |(\partial_t \Psi)\, v|.$$

By Young's inequality $2\,|(\partial_t \Psi)\, v| \leqslant \lambda^{-1} v^2 + \lambda(\partial_t \Psi)^2$. Using the Lipschitz continuity of the hyperbolic tangent we get

$$\|v(t)\|_2^2 \leqslant \|v(0)\|_2^2 + C\lambda^{-1}\,\|v\|^2_{L^2([0,t]\times\Omega)} + C\lambda^{-3}\,\|\Psi\|^2_{L^2([0,t]\times\Omega)} \leqslant C\lambda^{p-5}. \tag{3.12}$$

by (3.11)(i) and (iii). We improve the estimate to $L^\infty([0,T]\times\Omega)$ by rough estimates on the heat semigroup with periodic boundary conditions. For the required estimates on the semigroup we refer e.g., to ref. 27. Because of our choice of boundary conditions, we have to work with functions of average 0. We use the estimates already achieved together with the equation for u (1.2) in the mild sense, i.e., as integral equation. ("Variation of constants formula"), then we use the smoothing properties of the semigroup to get finally

$$\|u(t) - u_A(t)\|_\infty \leqslant C\lambda^{l(p,n)} \tag{3.13}$$

on $[0, T^*)$, where $l(p,n)$ is strictly increasing in p, so in particular $T^* = T_{\max} - \delta$ for p large enough and δ arbitrarily small, but fixed.

For the L^∞-convergence we use the variation of constants formula for m: Let $h_a := J^\lambda * m_a + \lambda u_a$, $h := J^\lambda * m + \lambda u$, then

$$\Psi(t) = e^{-\lambda^{-2} t}\Psi(0) + \int_0^t e^{-\lambda^{-2}(t-s)}\lambda^{-2}[\tanh(\beta h_a) - \tanh(\beta h)],$$

so for p large enough

$$|\Psi(t,x)| \leqslant C(T)\int_0^t (\lambda^{-2}(J^\lambda * |\Psi(s)|)(x) + \lambda^{-1}\,|u_a(s,x) - u(s,x)|)\,\mathrm{d}s = \mathcal{O}(\lambda).$$

4. APPENDIX

4.1. Existence and Regularity for Linear Nonlocal Equations

Most of the results in this chapter are contained in or rather easy corollaries of results from refs. 7, 14, 17, 19, and 21.

Definition 4.1. Denote by μ the measure $N(1-\bar{m}(r)^2)^{-1}\,dr$. Let for $f \in L^\infty \cup L^2(\mathbb{R})$

$$(L^{(1)}_{\bar{m}} f)(x) = -f(x) + \beta(1-\bar{m}(x)^2)(\bar{J} * f)(x).$$

Further define $\langle f, g\rangle_\mu := \int_{\mathbb{R}} f(x)\, g(x)[\beta(1-\bar{m}^2)]^{-1}\,dx$, and define the weighted L^∞-space $\|f\|_\delta = \sup_x e^{-\delta|x|}\,|f(x)|$. (See also refs. 17, p. 550.)

There is a spectral gap for $L^1_{\bar{m}}$ in L^2 and moreover if $\langle R, \bar{m}'\rangle_\mu = 0$, then there is an $a > 0$ s.t. $\|e^{L^{(1)}_{\bar{m}}t} R\|_\delta < e^{-at}\,\|R\|_\delta$.

Lemma 4.2. If $\langle R, \bar{m}'\rangle_\mu = 0$, then the following assertions hold (Fredholm situation):

(1) There is a unique solution f orthogonal to $\bar{m}'$ of $L^{(1)}_{\bar{m}} f = R$.

(2) Regularity: Moreover there is an $\alpha > 0$ s.t. for any δ with $-\alpha < \delta < \alpha$ we have the following regularity result: $\|f\|_\delta \leqslant C_\delta\,\|R\|_\delta$.

(3) If $\|\bar{m}'(x)^{-1}R(x)\|_\infty < \infty$, then $|f(x)| \leqslant C(1+|x|)\,\bar{m}'(x)$.

In particular if the right hand side decays exponentially, then the solution decays exponentially. This holds also for non-zero boundary conditions at $\pm\infty$:

Lemma 4.3. There is a $\delta_0 > 0$ such that if $L^{(1)}_{\bar{m}} f = R$, $\langle R, \bar{m}'\rangle_\mu = 0$ and

$$\|R(z) - c^+ 1_{z>0}(z) - c^- 1_{z<0}(z)\|_\delta < \infty,$$

(exponential convergence towards a constant) for $\delta < \delta_0$, then

$$\lim_{z\to\pm\infty} f(z) = \frac{c^\pm}{\beta(1-m_\beta^2)-1},$$

with exponential rate of convergence δ.

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